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Best approximation by integer-valued functions

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Abstract

Given an integer function f, the problem is to find its best uniform approximation from a set K of integer-valued bounded functions. Under certain conditions on K, the best extremal (maximal or minimal) approximation is identified. Furthermore, the operator mapping f to its extremal best approximation is shown to be Lipschitzian with some constant C or optimal Lipschitzian having the smallest C among all such operators. The results are applied to approximation problems.

Keywords: Approximation problem; Integer-valued function; Bounded function; Uniform norm; Best approximation; Lipschitzian selection operator; Quasi-convex function; Convex function; Isotone function; Monotone function; Majorant; Minorant

1. Introduction

Let S be any set and let B be the Banach space of real-valued bounded functions f on S equipped with the uniform norm $\|f\| = \sup\{|f(s)| : s \in S\}$. Let $D \subset B$ be the set of all integer-valued functions on S, and $K \subset D$ be any nonempty set. For f in D, let $\Delta(f)$ denote the infimum of $\|f - k\|$ for k in K. The problem considered is to find f' in K so that

$$\Delta(f) = \|f - f'\| = \inf\{\|f - k\| : k \in K\}. \tag{1.1}$$

Such an f' is called a best approximation to f from K. The set of all best approximations to f, denoted by A_f, is not necessarily singleton in general. A Lipschitzian selection operator (LSO) T is defined to

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be a selection operator which maps each f in D to an f' in A_f so that for some least number $C(T)$ the following holds:

$$\|T(f) - T(h)\| \leq C(T)\|f - h\| \qquad \text{for all } f, h \text{ in } D.$$

Such a T is called an optimal Lipschitzian selection operator (OLSO) if $C(T) \leq C(T')$ for all LSOs T'. In this article we obtain certain conditions on K so that best approximations and LSOs can be identified. We considered a similar problem on the space B of bounded functions in an earlier article [1]. However, the integer condition imposed on D in the present framework is more restrictive. It will be seen that some of the results of [1] can be extended to the present framework with some changes and modification of proofs. A class of related problems on the space of bounded or continuous functions but without the integer restriction is considered in [2]. Two integer restricted approximation problems are analyzed in [3, 4]. The significance of the integer restriction is explained in [4]. Because of this restriction, any nonempty subset of D is not convex unless it is a singleton. Hence, the classical methods of approximation theory such as those given in [5,6] cannot be applied directly in the present framework.

We state below three conditions on K. Depending upon the case under consideration, only a subset of these conditions will be imposed on K.

(i) If $k \in K$, then $k + p \in K$ for all integers p.
(ii) If $K' \subset K$ is a set of functions uniformly bounded above on S, then the function k', which is the pointwise supremum of functions in K', is in K.
(iii) If $K' \subset K$ is a set of functions uniformly bounded below on S, then the function k', which is the pointwise infimum of functions in K', is in K.

Another related problem of interest is the following. For f in D, let $K_f = \{k \in K : k \leq f\}$, and $\overline{\Delta}(f)$ be the infimum of $\|f - k\|$ for k in K_f. The problem is to find an f' in K_f so that

$$\overline{\Delta}(f) = \|f - f'\| = \inf\{\|f - k\| : k \in K_f\}. \tag{1.2}$$

We state our main results in the next section. There we give examples of problems for which the above three conditions apply.

2. Main results and applications

For a given f in D, let $K_f = \{k \in K : k \leq f\}$ as above, and, in addition, let $K'_f = \{k \in K : k \geq f\}$. If condition (i) holds for K, then both K_f and K'_f are nonempty. To see this, let $g \in K$. Then $g - \|f - g\| \leq f$. Since $\|f - g\|$ is an integer, by condition (i), $g - \|f - g\|$ is in K_f. A similar proof applies to K'_f. Now, for f in D, let

$$\overline{f}(s) = \sup\{k(s) : k \in K_f\}, \qquad s \in S.$$
$$\underline{f}(s) = \inf\{k(s) : k \in K'_f\}, \qquad s \in S.$$

Note that if K satisfies condition (ii) (respectively condition (iii)), then $\overline{f}$ (respectively $\underline{f}$) is in K. We have, obviously, $\overline{f} \leq f \leq \underline{f}$. These two functions, $\overline{f}$ and $\underline{f}$ are, respectively called the greatest K-minorant and smallest K-majorant of f.

Proposition 2.1. *Assume K satisfies conditions* (i) *and* (ii). *Then $\|\overline{f} - \overline{h}\| \leq \|f - h\|$ for all f, h in D. Similarly, if K satisfies conditions* (i) *and* (iii), *then $\|\underline{f} - \underline{h}\| \leq \|f - h\|$ for all f, h in D.* □

The proof of this proposition is similar to that of Proposition 2.2 of [1]. To prove the next theorem, we state the following result which holds in broad generality [7, p. 17].

$$|\Delta(f) - \Delta(h)| \le \|f - h\|. \tag{2.1}$$

We denote by $\lceil x \rceil$, the ceiling function of x, i.e., the smallest integer greater than or equal to x.

Theorem 2.1. *The following applies to Problem* (1.1).

(a) *Assume K satisfies conditions* (i) *and* (ii). *Then*

$$\Delta(f) = \lceil \|f - \overline{f}\|/2 \rceil, \tag{2.2}$$

and $f' = \overline{f} + \Delta(f)$ is the maximal best approximation to f. Moreover, if $f, h \in D$, then

$$\|f' - h'\| \le \|f - h\|, \qquad \text{if } \Delta(f) = \Delta(h), \tag{2.3}$$

and

$$\|f' - h'\| \le 2\|f - h\|. \tag{2.4}$$

The operator $T : D \to K$ defined by $T(f) = f'$ is a Lipschitzian selection operator with $C(T) = 2$.

(b) *Assume K satisfies conditions* (i) *and* (iii). *Then* (a) *holds with $\overline{f}$ replaced by $\underline{f}$ and $f' = \underline{f} - \Delta(f)$, which is the minimal best approximation to f.*

Proof. This is a modification of the proof of Proposition 3.1 of [1]. Let $g \in K$, and $g_0 = g - \|f - g\|$. Since $\|f - g\|$ is an integer, by condition (i) on K, we have that $g_0 \in K$. Now, $f \ge g_0$. Hence, $f \ge \overline{f} \ge g_0$. This shows that $f - \overline{f} \le f - g + \|f - g\|$ or $\|f - \overline{f}\|/2 \le \|f - g\|$. Since $\|f - g\|$ is an integer, we must have $\lceil \|f - \overline{f}\|/2 \rceil \le \|f - g\|$. Hence, $\lceil \|f - \overline{f}\|/2 \rceil \le \Delta(f)$. Again, since $\lceil \|f - \overline{f}\|/2 \rceil$ is an integer, by condition (i), $f' = \overline{f} + \lceil \|f - \overline{f}\|/2 \rceil$ is in K. It is easy to show that $\|f - f'\| \le \lceil \|f - \overline{f}\|/2 \rceil$. This establishes that f' is a best approximation and that (2.2) holds. Suppose now that g is any best approximation. Then $f \ge g - \Delta(f)$. Consequently, $f \ge \overline{f} \ge g - \Delta(f)$ and hence $f' \ge g$. Thus f' is the maximal best approximation.

Now let $f' = \overline{f} + \Delta(f)$ and $h' = \overline{h} + \Delta(h)$ be two best approximations to f and h respectively. Then,

$$\|f' - h'\| \le \|\overline{f} - \overline{h}\| + |\Delta(f) - \Delta(h)|.$$

From this inequality, (2.1), and Proposition 2.1, both (2.3) and (2.4) follow. By (2.4), we have $C(T) \le 2$. To show $C(T) = 2$, let K be the set of all integer convex functions on $S = [0, 1]$. Clearly, each function in K is constant on $(0, 1)$ with possible discontinuities at 0 and 1. Let $f(0) = -1$, $f(s) = 1$ on $(0, 1]$, and $h(s) = 0$ on $[0, 1]$. Then $f'(s) = 0$ on $[0, 1)$, $f'(1) = 2$, and $h'(s) = 0$ on $[0, 1]$ as may be easily verified. Consequently, $\|f - h\| = 1$ and $\|f' - h'\| = 2$. Hence $C(T) = 2$. The proof of part (b) is similar. □

Theorem 2.2. *The following applies to Problem* (1.2). *Assume K satisfies conditions* (i) *and* (ii). *Then $\overline{f}$ is the maximal best approximation to f and $\overline{\Delta}(f) = \|f - \overline{f}\| \le 2\Delta(f)$. The operator $T : D \to K$ defined by $T(f) = \overline{f}$ is the unique optimal Lipschitzian selection operator with $C(T) = 1$.*

Proof. The proof of Theorem 3.2 of [1] may be applied by letting the first constant c in that proof be a positive integer, say 1. The second constant $c = \|f - \overline{f}\|$ defined there is clearly an integer since both f and $\overline{f}$ are integer functions. Hence, $h = \overline{f} + c$ is in K by condition (i) since $\overline{f}$ is in K by condition (ii). The rest of the proof applies verbatim. □

We now consider some applications of the problem. A function k defined on a convex set $S \subset R^n$ is said to be quasi-convex if $k(\lambda s + (1 - \lambda)t) \leq \max\{k(s), k(t)\}$, for all s, t in S and all $0 \leq \lambda \leq 1$ [8]. If S is not convex, for example, when it is a finite set, we define k on S to be quasi-convex if there exists a quasi-convex function k' on the convex hull $\text{co}(S)$ of S whose restriction to S is k. It is easy to see that conditions (i) and (ii) hold for the set K of all integer quasi-convex functions on S. The results of Theorems 2.1(a) and 2.2 then apply. When S is finite, polynomial algorithms for computation of a best approximation can be developed by methods similar to those given in [9]. For our second example, we consider approximation by integer convex functions on a set S. In a manner analogous to the above, we may define a convex function on a domain S, which is possibly non-convex, by simply extending its usual definition for a convex domain to a non-convex S. Again, it is easy to verify that conditions (i) and (ii) hold for the set K of integer convex functions on S. Hence Theorems 2.1(a) and 2.2 apply. If S is convex then an integer convex function on S is necessarily constant in the relative interior of S and may have discontinuities at the points of the relative boundary. If S is not convex, for example, if it is finite, then the set of integer convex functions on S may include non-constant functions. For our third example, let S be a partially ordered set and K, the set of all integer isotone functions on S. For example, S is a rectangle in R^n with usual vector ordering. It is easy to show that conditions (i), (ii) and (iii) hold for K. Hence, both (a) and (b) of Theorem 2.1 apply. See [4] where stronger results are obtained for such problems on finite sets under weighted uniform norm. If S is a real interval, then K is the set of integer valued monotone non-decreasing functions on S. See [3] for a least squares approximation problem involving these functions.

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